A remark on the complex interpolation for families of Banach spaces

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Abstract

We show by explicit examples that the complex interpolation for families of Banach spaces is not stable under rearrangement of the given family on the boundary, although, by [CCRSW82], it is stable when the latter family takes only 2 values. In our examples, we can even assume that the family takes only 3 values, which is best possible. We also characterize all the transformations on $\mathbb{T}$ that are invariant for complex interpolation at 0, they are precisely the origin-preserving inner functions.

1 Introduction

This paper is a remark on the theory of complex interpolation for families of Banach spaces, developed by Coifman, Cwikel, Rochberg, Sagher and Weiss in [CCRSW82]. To avoid technical difficulties, we will concentrate on finite dimensional spaces.

Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disc with boundary $\mathbb{T} = \partial\mathbb{D}$. The normalised Lebesgue measure on $\mathbb{T}$ is denoted by m. By an interpolation family, we mean a measurable family of complex N-dimensional normed spaces $\{E_\gamma : \gamma \in \mathbb{T}\}$, i.e., E_γ is $\mathbb{C}^N$ equipped with norm $\|\cdot\|_\gamma$ and for each $x \in \mathbb{C}^N$, the function $\gamma \mapsto \|x\|_\gamma$ defined on $\mathbb{T}$ is measurable. We should also assume that $\int \log^+ \|x\|_\gamma dm(\gamma) < \infty$ for any $x \in \mathbb{C}^N$. By definition, the interpolated space at 0 is

$$E[0] := H^\infty(\mathbb{T}; \{E_\gamma\})/zH^\infty(\mathbb{T}; \{E_\gamma\}).$$

That is, for all $x \in \mathbb{C}^N$,

$$\|x\|_{E[0]} = \inf\left\{\operatorname*{ess\,sup}_{\gamma \in \mathbb{T}} \|f(\gamma)\|_{E_\gamma} \Big| f : \mathbb{T} \to \mathbb{C}^N \text{ analytic}, f(0) = x\right\}.$$

More generally, for any $z \in \mathbb{D}$, the interpolated space at z for the family $\{E_\gamma : \gamma \in \mathbb{T}\}$ is denoted by $E[z]$ or $\{E_\gamma : \gamma \in \mathbb{T}\}[z]$ whose norm is defined by:

$$\|x\|_{E[z]} = \inf\left\{\operatorname*{ess\,sup}_{\gamma \in \mathbb{T}} \|f(\gamma)\|_{E_\gamma} \Big| f : \mathbb{T} \to \mathbb{C}^N \text{ analytic}, f(z) = x\right\}.$$

It is well-known (see [CCRSW82, Prop. 2.4]) that in the above definition $\operatorname*{ess\,sup}_{\gamma \in \mathbb{T}} \|f(\gamma)\|_{E_\gamma}$ can be replaced by $\left(\int \|f(\gamma)\|_{E_\gamma}^p P_z(d\gamma)\right)^{1/p}$ for $0 < p < \infty$ or $\exp\left(\int \log \|f(\gamma)\|_{E_\gamma} P_z(d\gamma)\right)$ without changing the norm on $E[z]$, where $P_z(d\gamma)$ is the harmonic measure on $\mathbb{T}$ associated to z.

The goal of this paper is to investigate when the norm of the space $E[0]$ is invariant under a (measure preserving) rearrangement of the family $\{E_\gamma : \gamma \in \mathbb{T}\}$. A trivial example of such a rearrangement is a rotation on $\mathbb{T}$. But, as we will see, there are non trivial instances of this phenomenon. In particular, we recall the following well-known result.

Theorem 1.1. *[CCRSW82, Cor. 5.1] If $X_\gamma = Z_0$ for all $\gamma \in \Gamma_0$ and $X_\gamma = Z_1$ for all $\gamma \in \Gamma_1$, where Γ_0 and Γ_1 are disjoint measurable sets whose union is $\mathbb{T}$, then $X[0] = (Z_0, Z_1)_\theta$, where $\theta = m(\Gamma_1)$ and $(Z_0, Z_1)_\theta$ is the classical complex interpolation space for the pair (Z_0, Z_1).*

The key fact behind this theorem is the existence for any measurable partition $\Gamma_0 \cup \Gamma_1$ of the unit circle of an origin-preserving inner function taking Γ_0 to an arc of length $2\pi m(\Gamma_0)$ and Γ_1 to the complementary arc of length $2\pi m(\Gamma_1)$. For details, see the appendix. More generally, complex interpolation at 0 is stable under the rearrangements given by any inner function vanishing at 0.

Proposition 1.2. *Let $\varphi : \mathbb{D} \to \mathbb{C}$ be an origin-preserving inner function. Its boundary value is denoted again by $\varphi : \mathbb{T} \to \mathbb{T}$. Then for any interpolation family $\{E_\gamma : \gamma \in \mathbb{T}\}$, the canonical identity:*

$$\{E_\gamma : \gamma \in \mathbb{T}\}[0] = \{E_{\varphi(\gamma)} : \gamma \in \mathbb{T}\}[0]$$

is isometric.

Proof. The proof is routine, for details, see the last step in the proof of Theorem 1.1 in the appendix. $\square$

Thus Theorem 1.1 shows in particular that in the 2-valued case, the complex interpolation is stable under rearrangement. We show that in the general case, this is usually not true, even for families consisting only of 3 distinct Hilbert spaces.

Our method also yields a characterization of all the transformations on $\mathbb{T}$ that are invariant for complex interpolation at 0, they are precisely the origin-preserving inner functions.

2 An approximation formula

In this section, we first recall some results from [HL58, §5] and [HL61, §10, §11, §12] in the forms that will be convenient for us, and then deduce from them a useful formula.

Let $W : \mathbb{T} \to M_N$ be a measurable positive semi-definite $N \times N$-matrix valued function such that $\mathrm{tr}(W)$ is integrable. Such a function should be considered as a matrix weight. Without mentioning, all matrix weights in this paper satisfy: There exist $c, C > 0$ such that

$$(1) \qquad\qquad cI \le W(\gamma) \le CI \quad \text{ for } a.e.\, \gamma \in \mathbb{T}.$$

For such a matrix weight, let $L_W^2 = L^2(\mathbb{T}, W; S_2^N)$ be the set of functions $f : \mathbb{T} \to M_N$ for which

$$\|f\|_{L_W^2}^2 = \int \mathrm{tr}\Big(f(\gamma)^* W(\gamma) f(\gamma)\Big) dm(\gamma) < \infty.$$

Clearly, L_W^2 is a Hilbert space.

We will consider the two subspaces $H^2(W) \subset L_W^2$ and $H_0^2(W) \subset L_W^2$ defined as follows:

$$H^2(W) = \{f \in L_W^2 | \hat{f}(n) = 0, \forall n < 0\},$$

$$H_0^2(W) = \{f \in L_W^2 | \hat{f}(n) = 0, \forall n \le 0\}.$$

Given the assumption (1) on W, the identity map $Id : L_2(\mathbb{T}; S_2^N) \to L_W^2$ is an isomorphism:

$$(2) \qquad\qquad c^{1/2}\|f\|_{L^2(\mathbb{T}; S_2^N)} \le \|f\|_{L_W^2} \le C^{1/2}\|f\|_{L^2(\mathbb{T}; S_2^N)}.$$

Thus the restrictions $Id : H^2(\mathbb{T}; S_2^N) \to H^2(W)$ and $Id : H_0^2(\mathbb{T}; S_2^N) \to H_0^2(W)$ are also isomorphisms.

As usual, any element $f \in H^2(\mathbb{T}; S_2^N)$ will be identified with its holomorphic extension on $\mathbb{D}$.

We recall the following theorem from [HL58].

Theorem 2.1 (Helson-Lowdenslager). *Assume W a matrix weight as above. There exists $F \in H^2(\mathbb{T}; S_2^N)$ such that*

- $F(\gamma)^* F(\gamma) = W(\gamma)$ *for a.e. $\gamma \in \mathbb{T}$.*

- F *is a right outer function, that is, $F \cdot H^2(\mathbb{T}; S_2^N)$ is dense in $H^2(\mathbb{T}; S_2^N)$.*

Let Φ be the orthogonal projection of the constant function I to the subspace $H^2(W) \ominus H_0^2(W) \subset L_W^2$, i.e., $\Phi = P_{H^2(W) \ominus H_0^2(W)}(I)$, then

$$(3) \qquad \Phi(\gamma)^* W(\gamma) \Phi(\gamma) = |F(0)|^2 \ for \ a.e. \ \gamma \in \mathbb{T}.$$

Moreover, Φ and F and both invertible.

If F and G are two (right) outer functions such that

$$F(\gamma)^* F(\gamma) = G(\gamma)^* G(\gamma) = W(\gamma) \text{ for a.e. } \gamma \in \mathbb{T},$$

then there is a constant unitary matrix $U \in \mathscr{U}(N)$ such that $F(z) = UG(z)$ for all $z \in \mathbb{D}$. In particular, $|F(0)|^2 = |G(0)|^2$ is uniquely determined, as shown by the equation (3). Within all possible such outer functions, there is a unique one such that $F(0)$ is positive, we will denote it by F_W.

Let $\Psi = P_{H_0^2(W)}(I)$, where the orthogonal projection $P_{H_0^2(W)}$ is defined on the space L_W^2. Clearly, we have

$$(4) \qquad \Phi = I - \Psi.$$

We have already known that set theoretically, $H_0^2(W) = H_0^2(\mathbb{T}; S_2^N)$, and they are canonically isomorphic as Banach spaces, thus we have a Fourier series for $\Psi \in H_0^2(W) = H_0^2(\mathbb{T}; S_2^N)$:

$$\Psi = \sum_{n \geq 1} \widehat{\Psi}(n) \gamma^n;$$

where the convergence is a priori in $H_0^2(\mathbb{T}; S_2^N)$ and hence in $H_0^2(W)$.

By definition, Ψ is characterized by the following: for any $A \in M_N$ and any $n \geq 1$, we have $\langle \Psi, \gamma^n A \rangle_{L_W^2} = \langle I, \gamma^n A \rangle_{L_W^2}$, that is

$$\int \mathrm{tr}(\gamma^{-n} A^* W \Psi) dm(\gamma) = \int \mathrm{tr}(\gamma^{-n} A^* W) dm(\gamma), \text{ for any } A \in M_N \text{ and any } n \geq 1.$$

Hence

$$(5) \qquad \int \gamma^{-n} W \Psi dm(\gamma) = \int \gamma^{-n} W dm(\gamma), \text{ for } n \geq 1.$$

We denote by P_+ the orthogonal projection of $L^2(\mathbb{T})$ onto the subspace $H_0^2(\mathbb{T})$. The generalized projection $P_+ \otimes I_X$ on $L_p(\mathbb{T}; X)$ for $1 < p < \infty$ will still be denoted by P_+. Note that P_+ is slightly different to the usual Riesz projection, the latter is defined as the orthogonal projection onto $H^2(\mathbb{T})$. Similarly, we denote by P_- the orthogonal projection onto $\overline{H_0^2(\mathbb{T})}$ and also its generalisation on $L_p(\mathbb{T}; X)$ when it is bounded. With this notation, the equation system (5) is equivalent to

$$(6) \qquad P_+(W\Psi) = P_+(W).$$

Key observation: If W is a perturbation of identity, that is, if there exists a measurable function $\Delta : \mathbb{T} \to M_N$ such that

$$\Delta(\gamma)^* = \Delta(\gamma) \text{ for } a.e. \gamma \in \mathbb{T} \text{ and } \|\Delta\|_{L_\infty(\mathbb{T}; M_N)} < 1$$

and

$$W = I + \Delta;$$

then the equation (6) has the form

$$(7) \qquad \Psi + P_+(\Delta\Psi) = P_+(\Delta).$$

The above equation can be solved using a Taylor series.

To make the last sentence in the preceding observation rigorous, we introduce the following Toeplitz type operator:

$$T_\Delta : H_0^2(\mathbb{T}; S_2^N) \xrightarrow{L_\Delta} L^2(\mathbb{T}; S_2^N) \xrightarrow{P_+} H_0^2(\mathbb{T}; S_2^N);$$

where $L_\Delta : H_0^2(\mathbb{T}; S_2^N) \to L^2(\mathbb{T}; S_2^N)$ is the restriction of the left multiplication by Δ on the subspace $H_0^2(\mathbb{T}; S_2^N)$. More precisely,

$$(L_\Delta f)(\gamma) = \Delta(\gamma) f(\gamma) \text{ for any } f \in L^2(\mathbb{T}; S_2^N).$$

Clearly, we have

$$\|T_\Delta\| \le \|\Delta\|_{L_\infty(\mathbb{T}; M_N)} < 1.$$

The term $P_+(\Delta)$ in equation (7) should be treated as an element in $H_0^2(\mathbb{T}; S_2^N)$, then the equation (7) has the form

$$(8) \qquad (Id + T_\Delta)(\Psi) = P_+(\Delta).$$

Since $\|T_\Delta\| < 1$, the operator $Id + T_\Delta$ is invertible. Thus equation (8) has a unique solution $\Psi \in H_0^2(\mathbb{T}; S_2^N) = H_0^2(W)$ given by the formula:

$$(9) \qquad \Psi = (Id + T_\Delta)^{-1}(P_+(\Delta)) = \sum_{n=0}^\infty (-1)^n T_\Delta^n(P_+(\Delta));$$

where $T_\Delta^0(P_+(\Delta)) = P_+(\Delta)$, the convergence is understood in the space $H_0^2(\mathbb{T}; S_2^N)$. Combining equations (3), (4) et (9), we deduce the following formula:

$$|F_{I+\Delta}(0)|^2 = \left[I - \sum_{n=0}^\infty (-1)^n T_\Delta^n(P_+(\Delta)) \right]^* (I + \Delta) \times$$

$$\times \left[I - \sum_{n=0}^\infty (-1)^n T_\Delta^n(P_+(\Delta)) \right].$$

We summarize the above discussion in the following:

Proposition 2.2. *Let $\Delta : \mathbb{T} \to M_N$ be a measurable bounded selfadjoint function such that $\|\Delta\|_{L_\infty(\mathbb{T};M_N)} < 1$. Let $\varepsilon \in [0,1]$, then we have*

$$|F_{I+\varepsilon\Delta}(0)|^2 = \Big[I - \sum_{n=0}^{\infty}(-1)^n\varepsilon^{n+1}T_\Delta^n(P_+(\Delta))\Big]^*(I + \varepsilon\Delta)\times$$

$$\times\Big[I - \sum_{n=0}^{\infty}(-1)^n\varepsilon^{n+1}T_\Delta^n(P_+(\Delta))\Big].$$

In particular, we have

(10) $$|F_{I+\varepsilon\Delta}(0)|^2 = I + \varepsilon\widehat{\Delta}(0) - \varepsilon^2\sum_{n\geq 1}|\widehat{\Delta}(n)|^2 + \mathcal{O}(\varepsilon^3), \ \text{ as } \varepsilon \to 0^+.$$

Proof. It suffices to prove the approximation identity (10). We have

$$
\begin{aligned}
|F_{I+\varepsilon\Delta}(0)|^2 &= \Big[I - \varepsilon P_+(\Delta) + \varepsilon^2 T_\Delta(P_+(\Delta)) + \mathcal{O}(\varepsilon^3)\Big]^*(I + \varepsilon\Delta) \times \\
&\quad \times \Big[I - \varepsilon P_+(\Delta) + \varepsilon^2 T_\Delta(P_+(\Delta)) + \mathcal{O}(\varepsilon^3)\Big]
\end{aligned}
$$

(11) $$= I + \varepsilon R_1 + \varepsilon^2 R_2 + \mathcal{O}(\varepsilon^3), \text{ as } \varepsilon \to 0^+;$$

where

$$R_1 = \Delta - P_+(\Delta) - P_+(\Delta)^*,$$

$$R_2 = P_+(\Delta)^*P_+(\Delta) - \Delta P_+(\Delta) - P_+(\Delta)^*\Delta + T_\Delta(P_+(\Delta)) + T_\Delta(P_+(\Delta))^*.$$

For R_1, we note that since Δ is selfadjoint, $P_-(\Delta) = P_+(\Delta)^*$ and hence

(12) $$\Delta = P_+(\Delta) + P_+(\Delta)^* + \widehat{\Delta}(0).$$

Thus

$$R_1 = \widehat{\Delta}(0).$$

For R_2, we note that since the left hand side of equation (11) is independent of $\gamma \in \mathbb{T}$, the right hand side should also be independent of γ, hence R_2 must be independent of γ, it follows that

$$
\begin{aligned}
R_2 &= \int R_2(\gamma)dm(\gamma) \\
&= \int \Big(P_+(\Delta)^*P_+(\Delta) - \Delta P_+(\Delta) - P_+(\Delta)^*\Delta\Big)dm(\gamma) \\
&= -\sum_{n\geq 1}\widehat{\Delta}(n)^*\widehat{\Delta}(n) = -\sum_{n\geq 1}|\widehat{\Delta}(n)|^2.
\end{aligned}
$$

$\square$

Remark 2.3. *The following alternative verification of the identity*

$$R_2 = -\sum_{n\geq 1}|\widehat{\Delta}(n)|^2$$

might be helpful for the reader. Note that

$$-\Delta P_+(\Delta) + T_\Delta(P_+(\Delta)) = -\Delta P_+(\Delta) + P_+(\Delta P_+(\Delta)) = -P_{\leq 0}(\Delta P_+(\Delta)),$$

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where $P_{\leq 0}$ is the projection on the subspace of functions with spectrum supported on $\mathbb{Z}_{\leq 0}$. Since $P_{\leq 0}(P_{\geq 0}(\Delta)P_+(\Delta)) = 0$, we have

$$P_{\leq 0}(\Delta P_+(\Delta)) = P_{\leq 0}\Big(P_-(\Delta)P_+(\Delta)\Big) = P_{\leq 0}\Big(P_+(\Delta)^*P_+(\Delta)\Big).$$

Note also that

$$\Big(P_+(\Delta)^*P_+(\Delta)\Big)^{\wedge}(0) = \sum_{n \geq 1} |\widehat{\Delta}(n)|^2,$$

then it is clear that

$$-\Delta P_+(\Delta) - P_+(\Delta)^*\Delta + T_\Delta(P_+(\Delta)) + T_\Delta(P_+(\Delta))^*$$
$$= -P_{\leq 0}\Big(P_+(\Delta)^*P_+(\Delta)\Big) - P_{\leq 0}\Big(P_+(\Delta)^*P_+(\Delta)\Big)^*$$
$$= -P_-\Big(P_+(\Delta)^*P_+(\Delta)\Big) - P_-\Big(P_+(\Delta)^*P_+(\Delta)\Big)^* - 2\sum_{n \geq 1} |\widehat{\Delta}(n)|^2.$$

Since $P_+(\Delta)^*P_+(\Delta)$ is selfadjoint, we can use the identity

$$P_+\Big(P_+(\Delta)^*P_+(\Delta)\Big) = P_-\Big(P_+(\Delta)^*P_+(\Delta)\Big)^*$$

to get

$$P_+(\Delta)^*P_+(\Delta) = P_+\Big(P_-(\Delta)^*P_+(\Delta)\Big)^* + P_-\Big(P_+(\Delta)^*P_+(\Delta)\Big) + \sum_{n \geq 1} |\widehat{\Delta}(n)|^2.$$

Then it is easy to see that

$$R_2 = -\sum_{n \geq 1} |\widehat{\Delta}(n)|^2.$$

3   Interpolation family in continuous case

To any invertible matrix $A \in GL_N(\mathbb{C})$ is associated a Hilbertian norm $\|\cdot\|_A$ on $\mathbb{C}^N$, which is defined as follows:

$$\|x\|_A = \|Ax\|_{\ell_2^N}, \text{ for any } x \in \mathbb{C}^N;$$

where ℓ_2^N denotes the space $\mathbb{C}^N$ with the usual Euclidean norm.

Let us denote

$$\ell_A^2 := (\mathbb{C}^N, \|\cdot\|_A).$$

Clearly, we have the following elementary properties:

- Let $A, B \in GL_N(\mathbb{C})$, then they define the same norm on $\mathbb{C}^N$ if and only if $|A| = |B|$. Thus, if $U \in \mathscr{U}(N)$ is a $N \times N$ unitary matrix, then $\|\cdot\|_{UA} = \|\cdot\|_A$.

- We define a pairing $(x, y) = \sum_{n=1}^N x_n y_n$ for any $x, y \in \mathbb{C}^N$, then under this pairing, we have the canonical isometries:

$$(\ell_A^2)^* = \ell_{A^{-T}}^2;$$

where A^{-T} is the inverse of the tranpose matrix A^T.

- We have the following canonical isometries:

$$\overline{\ell_A^2} = \ell_{\overline{A}}^2 \text{ and } \overline{\ell_A^2}^* = \ell_{(A^*)^{-1}}^2.$$

Consider an $N \times N$-matrix weight W. To such a weight is associated an interpolation family

$$\{\ell^2_{w(\gamma)} : \gamma \in \mathbb{T}\}, \text{ where } w(\gamma) = \sqrt{W(\gamma)}.$$

The following elementary proposition will be used frequently:

Proposition 3.1. *For interpolation family $\{E_\gamma : \gamma \in \mathbb{T}\}$ with $E_\gamma = \ell^2_{w(\gamma)}$, we have $E[0] = \ell^2_{F(0)}$, that is,*

$$\|x\|_{E[0]} = \|F(0)x\|_{\ell^N_2}, \text{ for all } x \in \mathbb{C}^N;$$

where $F(z)$ is any right outer function associated to the weight W.

Proof. By the definition of right outer function associated to the weight W,

$$(13) \qquad\qquad F(\gamma)^* F(\gamma) = W(\gamma) \text{ for } a.e. \, \gamma \in \mathbb{T}.$$

For any $x \in \mathbb{C}^N$, define an analytic function $f_x : \mathbb{D} \to \mathbb{C}^N$ by

$$f_x(z) = F(z)^{-1} F(0) x,$$

then $f_x(0) = x$ and for a.e. $\gamma \in \mathbb{T}$,

$$
\begin{aligned}
\|f_x(\gamma)\|^2_{w(\gamma)} &= \langle W(\gamma) F(\gamma)^{-1} F(0) x, F(\gamma)^{-1} F(0) x \rangle \\
&= \langle F(\gamma)^* F(\gamma) F(\gamma)^{-1} F(0) x, F(\gamma)^{-1} F(0) x \rangle \\
&= \|F(0)x\|^2_{\ell^N_2}.
\end{aligned}
$$

This shows that $\|f_x\|_{H^\infty(\mathbb{T}; \{E_\gamma\})} \le \|F(0)x\|_{\ell^N_2}$, whence

$$\|x\|_{E[0]} \le \|F(0)x\|_{\ell^N_2} = \|x\|_{\ell^2_{F(0)}}.$$

The converse inequality will be given by duality, it suffices to show that

$$\|x\|_{E[0]^*} \le \|x\|_{\ell^2_{F(0)^{-T}}} = \|x\|_{(\ell^2_{F(0)})^*}.$$

Consider the dual interpolation family $\{E^*_\gamma : \gamma \in \mathbb{T}\} = \{\ell^2_{w(\gamma)^{-T}} : \gamma \in \mathbb{T}\}$, which is naturally given by the weight $W(\gamma)^{-T} = (w(\gamma)^{-T})^* w(\gamma)^{-T}$. By [CCRSW82, Th. 2.12], we have a canonical isometry

$$\{E^*_\gamma : \gamma \in \mathbb{T}\}[0] = E[0]^*.$$

The identity (13) implies

$$(F(\gamma)^{-T})^* F(\gamma)^{-T} = W(\gamma)^{-T} \text{ for } a.e. \, \gamma \in \mathbb{T}.$$

Thus $F(z)^{-T}$ is the right outer function associated to the weight $W(\gamma)^{-T}$. Then the same argument as above yields that

$$\|x\|_{E[0]^*} \le \|x\|_{\ell^2_{F(0)^{-T}}} = \|x\|_{(\ell^2_{F(0)})^*}.$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Remark 3.2. *More generally, assume that X is a (finite dimensional) normed space such that $M_N \subset \mathrm{End}(X)$ and $\|u \cdot x\|_X = \|x\|_X$ for any $u \in \mathscr{U}(N)$. For instance $X = S^N_p$ ($1 \le p \le \infty$) and M_N acts on S^N_p by the usual left multiplications of matrices. Consider the interpolation family $E_\gamma = (X, \|\cdot\|_{X; A(\gamma)})$ with $\|x\|_{X; A(\gamma)} = \|A(\gamma) \cdot x\|_X$ for any $\gamma \in \mathbb{T}$, then $E[0] = (X, \|\cdot\|_{B(0)})$ with $\|x\|_{B(0)} = \|B(0) \cdot x\|_X$, where $B(z)$ is any right outer function associated to the matrix weight $A(\gamma)^* A(\gamma)$.*

The following result is probably known to the experts of prediction theory, since we do not find it in the literature, we include its proof.

Proposition 3.3. *The function $\{W(\gamma) : \gamma \in \mathbb{T}\} \mapsto F_W(0)$ or equivalently $\{W(\gamma) : \gamma \in \mathbb{T}\} \mapsto |F(0)|^2$ is not stable under rearrangement. More precisely, there exists a family $\{W(\gamma) : \gamma \in \mathbb{T}\}$ and a measure preserving mapping $S : \mathbb{T} \to \mathbb{T}$, such that*

$$F_W(0) \neq F_{W \circ S}(0).$$

Before we proceed to the proof of the proposition, let us mention that if the weight $W(\gamma)$ takes only 2 distinct values, i.e., if $W(\gamma) = A_0$ for $\gamma \in \Gamma_0$ and $W(\gamma) = A_1$ for $\gamma \in \Gamma_1$ with $\mathbb{T} = \Gamma_0 \cup \Gamma_1$ a measurable partition, then a detailed computation shows that we have

$$F_W(0)^2 = A_0^{1/2}(A_0^{-1/2}A_1A_0^{-1/2})^{m(\Gamma_1)}A_0^{1/2} = A_1^{1/2}(A_1^{-1/2}A_0A_1^{-1/2})^{m(\Gamma_0)}A_1^{1/2}.$$

In particular, $F_W(0) = F_{W \circ M}(0)$ for any measure preserving mapping $M : \mathbb{T} \to \mathbb{T}$. Of course, this can be viewed as a special case of Theorem 1.1. The fact that we can calculate $F_W(0)$ efficiently in the above situation is due to the fundamental fact that two quadratic forms can always be simultaneously diagonalized.

Proof. Fix $r > 0$, define two M_2-valued bounded analytic functions $F_1, F_2 : \mathbb{D} \to M_2$ by

$$F_1(z) = \begin{bmatrix} (1+r^2)^{1/4} & r(1+r^2)^{-1/4}z \\ 0 & (1+r^2)^{-1/4} \end{bmatrix},$$

$$F_2(z) = \begin{bmatrix} (1+r^2)^{-1/4} & 0 \\ r(1+r^2)^{-1/4}z & (1+r^2)^{1/4} \end{bmatrix}.$$

Note that they are both outer since $z \to F_1(z)^{-1}$ and $z \to F_2(z)^{-1}$ are bounded on $\mathbb{D}$. By a direct computation,

$$F_1(e^{i\theta})^*F_1(e^{i\theta}) = W_1(e^{i\theta}) = \begin{bmatrix} (1+r^2)^{1/2} & re^{i\theta} \\ re^{-i\theta} & (1+r^2)^{1/2} \end{bmatrix},$$

$$F_2(e^{i\theta})^*F_2(e^{i\theta}) = W_2(e^{i\theta}) = \begin{bmatrix} (1+r^2)^{1/2} & re^{-i\theta} \\ re^{i\theta} & (1+r^2)^{1/2} \end{bmatrix}.$$

If we define $S : \mathbb{T} \to \mathbb{T}$ by $S(\gamma) = \overline{\gamma}$, then S is measure preserving and $W_2 = W_1 \circ S$. By noting that $F_1(0)$ and $F_2(0)$ are positive, we have $F_1 = F_{W_1}$ and $F_2 = F_{W_2} = F_{W_1 \circ S}$. However, $F_{W_1 \circ S}(0) = F_2(0) \neq F_1(0) = F_{W_1}(0)$. $\square$

We denote

$$W^{(r)}(e^{i\theta}) := \begin{bmatrix} (1+r^2)^{1/2} & re^{i\theta} \\ re^{-i\theta} & (1+r^2)^{1/2} \end{bmatrix},$$

and let $w^{(r)}(\gamma) = \sqrt{W^{(r)}(\gamma)}$. The notation $S : \mathbb{T} \to \mathbb{T}$ will be reserved for the complex conjugation mapping.

An immediate consequence of Propositions 3.1 and 3.3 is the following:

Corollary 3.4. *The interpolation family $\{\widetilde{E}_\gamma^{(r)} = \ell^2_{(w^{(r)} \circ S)(\gamma)} : \gamma \in \mathbb{T}\}$ is a rearrangement of the family $\{E_\gamma^{(r)} = \ell^2_{w^{(r)}(\gamma)} : \gamma \in \mathbb{T}\}$. The identity mapping $Id : \widetilde{E}^{(r)}[0] \to E^{(r)}[0]$ has norm*

$$\|Id : \widetilde{E}^{(r)}[0] \to E^{(r)}[0]\| = (1+r^2)^{1/2}.$$

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Proof. Indeed, we have:

$$\|Id : \widetilde{E}^{(r)}[0] \to E^{(r)}[0]\| = \sup_{x \neq 0} \frac{\|F_{W^{(r)}}(0)x\|_{\ell_2^2}}{\|F_{W^{(r)} \circ S}(0)x\|_{\ell_2^2}}$$
$$= \|F_{W^{(r)}}(0)F_{W^{(r)} \circ S}(0)^{-1}\|_{M_2} = (1 + r^2)^{1/2}.$$

$\square$

Remark 3.5. *By Corollary 3.4 and a suitable discretization argument, we can show that if* $J_k = \left\{ e^{i\theta} : \frac{(k-1)\pi}{4} \leq \theta < \frac{k\pi}{4} \right\}$, *for* $1 \leq k \leq 8$, *and let* $\gamma_k \in J_k$ *be the center point on* J_k, *then the interpolation families* $B_\gamma^{(r_0)} = \ell^2_{w^{(r_0)}(\gamma_k)}$ *if* $\gamma \in J_k$ *and* $\widetilde{B}_\gamma^{(r_0)} = \ell^2_{w^{(r_0)}(\bar{\gamma}_k)}$ *if* $\gamma \in J_k$ *for* $r_0 = \sqrt{2 + 2\sqrt{2}}$ *give different interpolation space at 0, i.e.,*

$$\|Id : \widetilde{B}^{(r_0)}[0] \to B^{(r_0)}[0]\| > 1.$$

We omit its proof, because in the next section, we give a better result by using the formula obtained in §2.

4 Interpolation for three Hilbert spaces

In this section, we will show that complex interpolation is not stable even for a familiy taking only 3 distinct Hilbertian spaces. The starting point of this section is Proposition 2. Our proof is somewhat abstract, but it explains why the 3-valued case becomes different from the 2-valued case, the idea used in the proof will be applied further to get a characterization of measurable transformations on $\mathbb{T}$ that perserve complex interpolation at 0.

Theorem 4.1. *There are two different measurable partitions of the unit circle:*

$$\mathbb{T} = S_1 \cup S_2 \cup S_3 = S_1' \cup S_2' \cup S_3', \ m(S_k) = m(S_k'), \ for \ k = 1, 2, 3,$$

and three constant selfadjoint matrices $\Delta_k \in M_2$ *for* $k = 1, 2, 3$, *such that if we let*

$$\Delta = \Delta_1 1_{S_1} + \Delta_2 1_{S_2} + \Delta_3 1_{S_3} \ and \ \Delta' = \Delta_1 1_{S_1'} + \Delta_2 1_{S_2'} + \Delta_3 1_{S_3'},$$

then

$$\sum_{n \geq 1} |\widehat{\Delta}(n)|^2 \neq \sum_{n \geq 1} |\widehat{\Delta'}(n)|^2.$$

Before turning to the proof of the above theorem, we state our main result.

Corollary 4.2. *Let* Δ, Δ' *be as in Theorem 4.1. For* $0 < \varepsilon < \frac{1}{\|\Delta\|_\infty}$, *we define two matrix weights which are perturbation of identity:*

$$W_\varepsilon = I + \varepsilon\Delta, \ W_\varepsilon' = I + \varepsilon\Delta'.$$

Denote w_ε *and* w_ε' *the square root of* W_ε *and* W_ε' *respectively. Then there exists* $\varepsilon_0 < 1$ *such that whenever* $0 < \varepsilon < \varepsilon_0$, *we have*

$$|F_{W_\varepsilon}(0)|^2 \neq |F_{W_\varepsilon'}(0)|^2.$$

Thus, whenever $0 < \varepsilon < \varepsilon_0$, *the following two interpolation families*

$$\{\ell^2_{w_\varepsilon(\gamma)} : \gamma \in \mathbb{T}\}, \quad \{\ell^2_{w_\varepsilon'(\gamma)} : \gamma \in \mathbb{T}\}$$

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have the same distribution and take only 3 distinct values. However, the interpolation spaces at 0 given by these two families are different:

$$\ell^2_{F_{W_\varepsilon(0)}} \neq \ell^2_{F_{W'_\varepsilon(0)}}.$$

Proof. This is an immediate corollary of Proposition 2.2 and Theorem 4.1. The last assertion follows from Proposition 3.1. $\square$

Remark 4.3. *We verify that in the two main cases where the interpolation is stable under rearrangement, the function $\Delta \mapsto \sum_{n\geq 1} |\hat{\Delta}(n)|^2$ is stable under rearrangement. Note first that we have the following matrix identity:*

$$\sum_{n\geq 1} |\hat{\Delta}(n)|^2 = \int |P_+(\Delta)|^2 dm.$$

- *2-valued case: If Δ is a 2-valued selfadjoint function, i.e, there is a measurable subset $A \subset \mathbb{T}$ and two selfadjoint matrices $\Delta_1, \Delta_2 \in M_N$, such that $\Delta = \Delta_1 1_A + \Delta_2 1_{A^c}$ then*

$$\begin{aligned}
\sum_{n\geq 1} |\hat{\Delta}(n)|^2 &= \int |P_+(\Delta)|^2 dm = \int \left| P_+\left((\Delta_1 - \Delta_2)1_A + \Delta_2\right) \right|^2 dm \\
&= |\Delta_1 - \Delta_2|^2 \int |P_+(1_A)|^2 dm \\
&= \frac{m(A) - m(A)^2}{2} |\Delta_1 - \Delta_2|^2,
\end{aligned}$$

which depends on the measure of A but not the other structure of A.

More generally, we note in passing that for any real valued f in $L_2(\mathbb{T})$ the expression $2\|P_+(f)\|_2^2 = 2\sum_{n\geq 1} |\hat{f}(n)|^2$ coincides with the variance of f.

- *Rearrangement under inner functions: Let $\varphi : \mathbb{T} \to \mathbb{T}$ be the boundary value of an origin-preserving inner function. Assume $\Delta : \mathbb{T} \to M_N$ selfadjoint. Note that $P_+(\Delta \circ \varphi) = P_+(\Delta) \circ \varphi$ and that φ preserves the measure m. Hence*

$$\begin{aligned}
\sum_{n\geq 1} |\widehat{(\Delta \circ \varphi)}(n)|^2 &= \int |P_+(\Delta \circ \varphi)|^2 dm = \int |P_+(\Delta) \circ \varphi|^2 dm \\
&= \int |P_+(\Delta)|^2 dm = \sum_{n\geq 1} |\hat{\Delta}(n)|^2.
\end{aligned}$$

Proof of Theorem 4.1. Assume by contradiction that for any pair of 3-valued selfadjoint functions Δ and Δ' as in the statement of Theorem 4.1, we have

$$\tag{14} \sum_{n\geq 1} |\hat{\Delta}(n)|^2 = \sum_{n\geq 1} |\hat{\Delta'}(n)|^2.$$

We make the following reduction.

Step 1: The above assumption implies that for any pair of 3-valued matrix functions $\Delta, \Delta' \in L_\infty(\mathbb{T}; M_N)$ with having the same distribution $\Delta \stackrel{d}{=} \Delta'$, the equation (14) holds as well. Indeed, given such a pair, we can consider the pair of selfadjoint functions which are still 3-valued:

$$\gamma \to \begin{bmatrix} 0 & \Delta(\gamma)^* \\ \Delta(\gamma) & 0 \end{bmatrix} \text{ and } \gamma \to \begin{bmatrix} 0 & \Delta'(\gamma)^* \\ \Delta'(\gamma) & 0 \end{bmatrix}.$$

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Then the square of the n-th Fourier coefficient becomes

$$\begin{bmatrix} |\widehat{\Delta}(n)|^2 & 0 \\ 0 & |\widehat{\Delta^*}(n)|^2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} |\widehat{\Delta'}(n)|^2 & 0 \\ 0 & |\widehat{\Delta'^*}(n)|^2 \end{bmatrix}$$

respectively. The block $(1, 1)$-term of then gives the desired equation.

Step 2: If we take $N = 1$ in the above step, then the conclusion is that for any pair of 3-valued scalar functions $f, f' \in L_\infty(\mathbb{T})$ such that $f \overset{d}{=} f'$, we have $\sum_{n \geq 1} |\widehat{f}(n)|^2 = \sum_{n \geq 1} |\widehat{f'}(n)|^2$, or equivalently,

$$\|P_+(f)\|_2^2 = \|P_+(f')\|_2^2.$$

Consequence I: Under the above assumption, if (A_1, A_2) is a pair of two disjoint measurable subsets of $\mathbb{T}$, and (A_1', A_2') is another such pair such that $m(A_1) = m(A_1')$ and $m(A_2) = m(A_2')$, then

(15) $$\langle P_+(1_{A_1}), P_+(1_{A_2}) \rangle_{L^2(\mathbb{T})} = \langle P_+(1_{A_1'}), P_+(1_{A_2'}) \rangle_{L^2(\mathbb{T})}$$

Indeed, if we define $A_3 := \mathbb{T} \setminus (A_1 \cup A_2)$ and $A_3' := \mathbb{T} \setminus (A_1' \cup A_2')$. For any $\alpha \in \mathbb{C}, \alpha \neq 0, 1$, consider

$$f_\alpha = \alpha 1_{A_1} + 1_{A_2} + 0 \times 1_{A_3}, \quad f_\alpha' = \alpha 1_{A_1'} + 1_{A_2'} + 0 \times 1_{A_3'},$$

then f_α and f_α' are two functions taking exactly 3 values 0, 1, α and $f_\alpha \overset{d}{=} f_\alpha'$. Hence by the assumption, we have

(16) $$\|\alpha P_+(1_{A_1}) + P_+(1_{A_2})\|_2^2 = \|\alpha P_+(1_{A_1'}) + P_+(1_{A_2'})\|_2^2, \text{ for any } \alpha \in \mathbb{C}.$$

Note that for any measurable set A, since 1_A is real,

(17) $$\|P_+(1_A)\|_2^2 = \frac{m(A) - m(A)^2}{2}$$

Taking this in consideration, the equation (16) implies that

$$\Re\Big(\alpha\langle P_+(1_{A_1}), P_+(1_{A_2})\rangle\Big) = \Re\Big(\alpha\langle P_+(1_{A_1'}), P_+(1_{A_2'})\rangle\Big), \text{ for any } \alpha \in \mathbb{C},$$

hence the equation (15) holds.

Step 3: We can deduce from our assumption the following consequence.

Consequence II: For any pair of scalar functions $f, f' \in L_\infty(\mathbb{T})$ (without the assumption that they are both 3-valued), such that $f \overset{d}{=} f'$, we have $\|P_+(f)\|_2 = \|P_+(f')\|_2$.

Indeed, if

$$f = \sum_{k=1}^{n} f_k 1_{A_k}, \quad f' = \sum_{k=1}^{n} f_k 1_{A_k'},$$

where $(A_k)_{k=1}^n$ are disjoint subsets of $\mathbb{T}$, so is $(A_k')_{k=1}^n$, moreover $m(A_k) = m(A_k')$. By (15) and (17), we have

$$\begin{aligned} \|P_+(f)\|_2^2 &= \sum_{k=1}^{n} |f_k|^2 \cdot \|P_+(1_{A_k})\|_2^2 + \sum_{1 \leq k \neq l \leq n} f_k f_l \langle P_+(1_{A_1}), P_+(1_{A_2})\rangle \\ &= \sum_{k=1}^{n} |f_k|^2 \cdot \|P_+(1_{A_k'})\|_2^2 + \sum_{1 \leq k \neq l \leq n} f_k f_l \langle P_+(1_{A_1'}), P_+(1_{A_2'})\rangle \\ &= \|P_+(f')\|_2^2. \end{aligned}$$

Then by an approximation argument, more precisely, by using the fact that two functions $f, f' \in L^2(\mathbb{T})$ such that $f \stackrel{d}{=} f$ can be approximated in $L^2(\mathbb{T})$ by two sequences of simple functions (g_n) and (g'_n) such that $g_n \stackrel{d}{=} g'_n$, we can extend the above equality for pairs of equidistributed simple functions to the general equidistributed pairs of functions, as stated in Consequence II.

Step 4: Now if we take $f, f' \in L_\infty(\mathbb{T})$ to be $f(\gamma) = \gamma$ and $f'(\gamma) = \overline{\gamma}$, then $f \stackrel{d}{=} f'$, but we have $\|P_+(f)\|_2 = 1 \neq 0 = \|P_+(f')\|_2$, which contradicts Consequence II. This completes the proof. $\qquad \square$

Define
$$T_k := \left\{ e^{i\theta} \Big| \frac{2(k-1)\pi}{3} \leq \theta < \frac{2k\pi}{3} \right\} \text{ for } k = 1, 2, 3.$$

We claim that in Theorem 4.1 and hence in Corollary 4.2, we can take for example
$$S_1 = S'_1 = T_1, \quad S_2 = S'_3 = T_2, \quad S_3 = S'_2 = T_3.$$

Indeed, by the proof of Theorem 4.1, here we only need to show that
$$\langle P_+(1_{T_1}), P_+(1_{T_2}) \rangle \neq \langle P_+(1_{T_1}), P_+(1_{T_3}) \rangle.$$

Since $1_{T_1}(\gamma) = 1_{T_3}(e^{-i\frac{2\pi}{3}}\gamma)$ and $1_{T_2}(\gamma) = 1_{T_1}(e^{-i\frac{2\pi}{3}}\gamma)$, we have
$$\langle P_+(1_{T_1}), P_+(1_{T_3}) \rangle = \langle P_+(1_{T_2}), P_+(1_{T_1}) \rangle.$$

Thus we only need to show that
$$\text{(18)} \qquad \qquad \Im\Big(\langle P_+(1_{T_1}), P_+(1_{T_2}) \rangle \Big) \neq 0.$$

Note that
$$\Im\Big(\widehat{1_{T_1}}(n) \overline{\widehat{1_{T_2}}(n)} \Big) = \frac{\sin\frac{2\pi}{3}(1 - \cos\frac{2\pi}{3})}{2\pi^2 n^2} \times \begin{cases} 0, & \text{if } n \equiv 0 \mod 3; \\ 1, & \text{if } n \equiv 1 \mod 3; \\ -1, & \text{if } n \equiv 2 \mod 3. \end{cases}$$

Hence
$$\Im\Big(\langle P_+(1_{T_1}), P_+(1_{T_2}) \rangle \Big) = \frac{3\sin\frac{2\pi}{3}(1 - \cos\frac{2\pi}{3})}{2\pi^2} \sum_{k=0}^{\infty} \frac{2k+1}{(3k+1)^2(3k+2)^2},$$

which is non-zero, as we expected.

The same idea as in the proof of Theorem 4.1 yields the following somewhat interesting result. Combining with Proposition 1.2, we have characterized all measurable transformations on $\mathbb{T}$ that preserve complex interpolation at 0. At this stage, the proof is quite direct.

Theorem 4.4. *Let $\Theta : \mathbb{T} \to \mathbb{T}$ be a measurable transformation. If for any interpolation family $\{E_\gamma; \gamma \in \mathbb{T}\}$, we have*
$$\{E_\gamma : \gamma \in \mathbb{T}\}[0] = \{E_{\Theta(\gamma)} : \gamma \in \mathbb{T}\}[0],$$
then Θ is an origin-preserving inner function.

Proof. It suffices to show that $\Theta \in H_0^\infty(\mathbb{T})$, since by definition $\Theta(\gamma)$ has modulus 1 for *a.e.* $\gamma \in \mathbb{T}$. By Propositions 2.2, 3.1 and similar arguments in the proof of Theorem 4.1, we have

(19) $$\|P_+(f \circ \Theta)\|_2 = \|P_+(f)\|_2, \text{ for any scalar function } f \in L_\infty(\mathbb{T}).$$

Now take $f(\gamma) = \overline{\gamma}$, we have $\|P_+(\overline{\Theta})\|_2 = \|P_+(\overline{\gamma})\|_2 = 0$, which implies that $\overline{\Theta} \in \overline{H^\infty(\mathbb{T})}$ and hence $\Theta \in H^\infty(\mathbb{T})$. Then we can write $\Theta = \widehat{\Theta}(0) + P_+(\Theta)$. In (19), if we take $f(\gamma) = \gamma$, then $\|P_+(\Theta)\|_2 = \|P_+(\gamma)\|_2 = 1$. Note that

$$1 = \|\Theta\|_2^2 = |\widehat{\Theta}(0)|^2 + \|P_+(\Theta)\|_2^2,$$

whence $\widehat{\Theta}(0) = 0$. This completes the proof. $\square$

5   Some related comments

Recall that an N-dimensional (complex) Banach space $\mathscr{L}$ is called a (complex) Banach lattice with respect to a fixed basis $(e_1, \cdots, e_N)$ of $\mathscr{L}$ if it satisfies the lattice axiom: For any $x_k, y_k \in \mathbb{C}$ such that $|x_k| \leq |y_k|$ for all $1 \leq k \leq N$,

$$\|\sum_{k=1}^N x_k e_k\|_{\mathscr{L}} \leq \|\sum_{k=1}^N y_k e_k\|_{\mathscr{L}}.$$

Thus in particular,

$$\|\sum_{k=1}^N x_k e_k\|_{\mathscr{L}} = \|\sum_{k=1}^N |x_k| e_k\|_{\mathscr{L}}.$$

The above fixed basis $(e_1, \cdots, e_N)$ will be called a lattice-basis of $\mathscr{L}$. Such a Banach lattice $\mathscr{L}$ will be viewed as function spaces over the N-point set $[N] = \{1, \cdots, N\}$ in such a way that e_k corresponds to the Dirac function at the point k. Thus for $x, y \in \mathscr{L}$, we can write $|x| \leq |y|$ if $|x_k| \leq |y_k|$ for all $1 \leq k \leq N$, and $\log|x| = \sum_{k=1}^N \log|x_k| e_k$, suppose that $x_k \neq 0$ for all $1 \leq k \leq N$.

We will call $\{\mathscr{L}_\gamma = (\mathbb{C}^N, \|\cdot\|_\gamma) : \gamma \in \mathbb{T}\}$ a family of *compatible* Banach lattices, if there is an algebraic basis $(e_1, \cdots, e_N)$ of $\mathbb{C}^N$ which is simultaneously a lattice-basis of $\mathscr{L}_\gamma$ for a.e. $\gamma \in \mathbb{T}$ and such that

(20) $$0 < \underset{\gamma \in \mathbb{T}}{\text{ess inf}} \|e_k\|_\gamma \leq \underset{\gamma \in \mathbb{T}}{\text{ess sup}} \|e_k\|_\gamma < \infty \text{ for all } 1 \leq k \leq N.$$

In the sequel, the notation $\{\mathscr{L}_\gamma = (\mathbb{C}^N, \|\cdot\|_\gamma) : \gamma \in \mathbb{T}\}$ is reserved for a family of compatible Banach lattices with respect to the *canonical basis* of $\mathbb{C}^N$.

Complex interpolation at 0 for families of compatible Banach lattices is stable under any rearrangement. The proof of the following proposition is standard.

Proposition 5.1. *If $\{\mathscr{L}_\gamma = (\mathbb{C}^N, \|\cdot\|_\gamma) : \gamma \in \mathbb{T}\}$ be an interpolation family of compatible Banach lattices, then*

(21) $$\log \|x\|_{\mathscr{L}[0]} = \inf \int \log \|f(\gamma)\|_\gamma \, dm(\gamma),$$

where the infimum runs over the set of all measurable coordinate bounded functions $f : \mathbb{T} \to \mathbb{C}^N$, i.e., $f_k : \mathbb{T} \to \mathbb{C}$ is bounded for all $1 \leq k \leq N$ such that (by convention $\log 0 := -\infty$)

$$\log|x| \leq \int \log|f(\gamma)| \, dm(\gamma).$$

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In particular, if $M : \mathbb{T} \to \mathbb{T}$ is measure preserving and let $\{\widetilde{\mathscr{L}_\gamma} = \mathscr{L}_{M(\gamma)} : \gamma \in \mathbb{T}\}$, then

$$Id : \mathscr{L}[0] \to \widetilde{\mathscr{L}}[0]$$

is isometric.

Proof. It suffices to show (21). Assume that $x \in \mathbb{C}^N$ and $\|x\|_{\mathscr{L}[0]} < \lambda$. Without loss of generality, we can assume $x_k \neq 0$ for all $1 \leq k \leq N$. By the definition of $\mathscr{L}[0]$ there exists an analytic function $f = (f_1, \cdots, f_N) : \mathbb{D} \to \mathbb{C}^N$ such that

$$f(0) = x \text{ and } \operatorname*{ess\,sup}_{\gamma \in \mathbb{T}} \|f(\gamma)\|_\gamma < \lambda.$$

By (20), this implies in particular that f is coordinate bounded. Since $z \mapsto \log|f_k(z)|$ is subharmonic, we have

$$\log|x_k| = \log|f_k(0)| \leq \int \log|f_k(\gamma)|dm(\gamma), \text{ for } 1 \leq k \leq N.$$

Hence $\log|x| \leq \int \log|f(\gamma)|dm(\gamma)$. Obviously, $\int \log\|f(\gamma)\|_\gamma dm(\gamma) < \log\lambda$, whence

$$\inf \int \log\|f(\gamma)\|_\gamma \, dm(\gamma) \leq \log\|x\|_{\mathscr{L}[0]}.$$

Conversely, assume that $x \in \mathbb{C}^N$ and $x_k \neq 0$ for all $1 \leq k \leq N$ and let $f : \mathbb{D} \to \mathbb{C}^N$ be any coordinate bounded analytic function such that $\log|x| \leq \int \log|f(\gamma)|dm(\gamma)$. Then by (20), $\operatorname*{ess\,sup}_{\gamma \in \mathbb{T}} \|f(\gamma)\|_\gamma < \infty$ and there exists $y \in \mathbb{C}^N$ such that

(22)
$$|x| \leq |y| \text{ and } \log|y| = \int \log|f(\gamma)|dm(\gamma).$$

Define $u(\gamma) := \log|f(\gamma)|$. By assumption, $x_k \neq 0$ and f_k is bounded, hence $\log|f_k| \in L_1(\mathbb{T})$, so we can define the Hilbert transform of u_k. Let $\tilde{u}(\gamma)$ be the Hilbert transform of $u(\gamma)$ and define $g(\gamma) = e^{u(\gamma)+i\tilde{u}(\gamma)}$. Then $g_k(\gamma) = e^{u_k(\gamma)+i\tilde{u}_k(\gamma)}$ is the boundary value of an outer function, hence

$$\log|g_k(0)| = \int \log|g_k(\gamma)|dm(t) = \int u_k(\gamma)dm(\gamma) = \log|y_k|.$$

Thus $|y| = |g(0)|$. By [CCRSW82, Prop. 2.4], we have

$$
\begin{aligned}
\|y\|_{\mathscr{L}[0]} &= \|g(0)\|_{\mathscr{L}[0]} \leq \exp\left(\int \log\|g(\gamma)\|_\gamma \, dm(\gamma)\right) \\
&= \exp\left(\int \log\|f(\gamma)\|_\gamma \, dm(t)\right).
\end{aligned}
$$

It is easy to see that $\mathscr{L}[0]$ is a Banach lattice and by (22),

$$\|x\|_{\mathscr{L}[0]} \leq \|y\|_{\mathscr{L}[0]}.$$

Thus

$$\log\|x\|_{\mathscr{L}[0]} \leq \log\|y\|_{\mathscr{L}[0]} \leq \int \log\|f(\gamma)\|_\gamma \, dm(\gamma).$$

This proves the converse inequality. $\qquad\square$

Remark 5.2. *The preceding result should be compared with [CCRSW82, Cor. 5.2], where it is shown that*

$$\left\{ L^{p_\gamma}(X, \Sigma, \mu) : \gamma \in \mathbb{T} \right\}[z] = L^{p_z}(X, \Sigma, \mu),$$

where $1/p_z = \int (1/p_\gamma) P_z(d\gamma).$

If the Banach lattices $\mathscr{L}_\gamma$ considered above are all symmetric, i.e., for any permutation $\sigma \in \mathfrak{S}_N$ and any $x_k \in \mathbb{C}$,

$$\| \sum_{k=1}^{N} x_k e_{\sigma(k)} \|_{\mathscr{L}_\gamma} = \| \sum_{k=1}^{N} x_k e_k \|_{\mathscr{L}_\gamma},$$

then to each $\mathscr{L}_\gamma$ is associated a Schatten type space $S_{\mathscr{L}_\gamma} = (M_N, \| \cdot \|_{S_{\mathscr{L}_\gamma}}).$

The following proposition is classical (c.f. [Pie71]), we omit its proof.

Proposition 5.3. *Let* $\{\mathscr{L}_\gamma = (\mathbb{C}^N, \| \cdot \|_\gamma) : \gamma \in \mathbb{T}\}$ *be an interpolation family of compatible symmetric Banach lattices and consider the associated interpolation family:*

$$\{ S_{\mathscr{L}_\gamma} = (M_N, \| \cdot \|_{S_{\mathscr{L}_\gamma}}) : \gamma \in \mathbb{T} \}.$$

Then for any $z \in \mathbb{D}$, *we have the following isometric identification*

$$Id : S_{\mathscr{L}[z]} \to \{ S_{\mathscr{L}_\gamma} \}[z].$$

Combining Propositions 5.1 and 5.3, we have the following:

Corollary 5.4. *Consider the interpolation family* $\{S_{\mathscr{L}_\gamma} : \gamma \in \mathbb{T}\}$. *Let* $M : \mathbb{T} \to \mathbb{T}$ *be measure preserving and let* $\{\widetilde{S}_{\mathscr{L}_\gamma} = S_{\mathscr{L}_{M(\gamma)}} : \gamma \in \mathbb{T}\}$, *then*

$$Id : \{S_{\mathscr{L}_\gamma}\}[0] \to \{\widetilde{S}_{\mathscr{L}_\gamma}\}[0]$$

is isometric.

The following proposition is related to our problem, see the discussion after it.

Proposition 5.5. *Let* $\{E_\gamma : \gamma \in \mathbb{T}\}$ *be an interpolation family of* N-*dimensional spaces such that there exist* $c, C > 0$, *for any* $x \in \mathbb{C}^N$,

$$c \cdot \min_k |x_k| \leq \|x\|_\gamma \leq C \cdot \max_k |x_k| \text{ for a.e. } \gamma \in \mathbb{T}.$$

Assume that $Id : E_{\bar{\gamma}} \to \overline{E_\gamma}^*$ *is isometric for a.e.* $\gamma \in \mathbb{T}$. *Then*

$$E[\zeta] = \ell_2^N, \text{ for any } \zeta \in (-1, 1).$$

Proof. Fix $\zeta \in (-1, 1)$. For any $x \in \mathbb{C}^N$. Given any analytic function $f : \mathbb{D} \to \mathbb{C}^N$ such that $f(\zeta) = x$ and $\|f\|_{H^\infty(\{E_\gamma\})} < \infty$. Since $\zeta = \bar{\zeta}$, we have $f(\zeta) = f(\bar{\zeta}) = x$. The assumption on the interpolation family implies taht the function $z \mapsto \langle f(z), f(\bar{z}) \rangle$ is bounded analytic, hence

$$
\begin{aligned}
\log \|x\|_{\ell_2^N}^2 &= \log |\langle f(\zeta), f(\bar{\zeta}) \rangle| \leq \int \log |\langle f(\gamma), f(\bar{\gamma}) \rangle| P_\zeta(d\gamma) \\
&\leq \int \log \left(\|f(\gamma)\|_{E_\gamma} \|f(\bar{\gamma})\|_{\overline{E_\gamma^*}} \right) P_\zeta(d\gamma) \\
&= \int \log \left(\|f(\gamma)\|_{E_\gamma} \|f(\bar{\gamma})\|_{E_{\bar{\gamma}}} \right) P_\zeta(d\gamma) \\
&\leq \log \left(\|f\|_{H^\infty(\{E_\gamma\})}^2 \right).
\end{aligned}
$$

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Hence $\|x\|_{\ell_2^N} \leq \|f\|_{H^\infty(\{E_\gamma\})}$. It follows that

$$\|x\|_{\ell_2^N} \leq \|x\|_{E[\zeta]}.$$

By duality, this inequality also holds in the dual case, hence we must have $\|x\|_{\ell_2^N} = \|x\|_{E[\zeta]}$. $\qquad\square$

Let Q_j be the open arc of $\mathbb{T}$ in the j-th quadrant, i.e.,

$$Q_j = \left\{ e^{i\theta} : \frac{(k-1)\pi}{2} < \theta < \frac{k\pi}{2} \right\} \text{ for } 1 \leq j \leq 4.$$

Suppose that X and Y are N-dimensional, define two interpolation families $\{Z_\gamma : \gamma \in \mathbb{T}\}$ and $\{\widetilde{Z}_\gamma : \gamma \in \mathbb{T}\}$ by letting

$$Z_\gamma = \left\{ \begin{array}{ll} X, & \gamma \in Q_1 \\ Y, & \gamma \in Q_2 \\ \overline{Y^*}, & \gamma \in Q_3 \\ \overline{X^*}, & \gamma \in Q_4 \end{array} \right. , \qquad \widetilde{Z}_\gamma = \left\{ \begin{array}{ll} X, & \gamma \in Q_1 \\ Y, & \gamma \in Q_2 \\ \overline{X^*}, & \gamma \in Q_3 \\ \overline{Y^*}, & \gamma \in Q_4 \end{array} \right. .$$

By Proposition 5.5, $Z[0] = \ell_2^N$. For suitable choices of X and Y, we could have $\widetilde{Z}[0] \neq \ell_2^N$. More precisely, we have the following proposition.

Proposition 5.6. *For any $\alpha \in \mathbb{T}$, define a 2×2 selfadjoint matrix*

$$\delta_\alpha := \left[\begin{array}{cc} 0 & \overline{\alpha} \\ \alpha & 0 \end{array} \right].$$

For $0 < \varepsilon < 1$, let $w^{\alpha,\varepsilon} = (I + \varepsilon\delta_\alpha)^{1/2}$ and $X = \ell_{w^{\alpha,\varepsilon}}^2$. Consider the weight $W^{\alpha,\varepsilon}$ and the interpolation family generated by it as follows:

$$W^{\alpha,\varepsilon}(\gamma) = \left\{ \begin{array}{ll} I + \varepsilon\delta_\alpha, & \gamma \in Q_1 \\ (I + \varepsilon\overline{\delta_\alpha})^{-1}, & \gamma \in Q_2 \\ (I + \varepsilon\delta_\alpha)^{-1}, & \gamma \in Q_3 \\ I + \varepsilon\overline{\delta_\alpha}, & \gamma \in Q_4 \end{array} \right. ; \qquad \widetilde{Z_\gamma^{\alpha,\varepsilon}} = \left\{ \begin{array}{ll} X, & \gamma \in Q_1 \\ X^*, & \gamma \in Q_2 \\ \overline{X}^*, & \gamma \in Q_3 \\ \overline{X}, & \gamma \in Q_4 \end{array} \right. .$$

There exists $\alpha \in \mathbb{T}$ and $0 < \varepsilon_0 < 1$, such that if $0 < \varepsilon < \varepsilon_0$ then

$$\widetilde{Z^{\alpha,\varepsilon}}[0] \neq \ell_2^N.$$

Proof. We have

$$W^{\alpha,\varepsilon}(\gamma) = \left\{ \begin{array}{ll} I + \varepsilon\delta_\alpha, & \gamma \in Q_1 \\ I - \varepsilon\overline{\delta_\alpha} + \varepsilon^2 I + \mathcal{O}(\varepsilon^3), & \gamma \in Q_2 \\ I - \varepsilon\delta_\alpha + \varepsilon^2 I + \mathcal{O}(\varepsilon^3), & \gamma \in Q_3 \\ I + \varepsilon\overline{\delta_\alpha}, & \gamma \in Q_4 \end{array} \right. .$$

Applying a slightly modified variant of the approximation equation (10), we have

$$|F_{W^{\alpha,\varepsilon}}(0)|^2 = I + \frac{\varepsilon^2 I}{2} - \varepsilon^2 \left[\begin{array}{cc} \|P_+(h_\alpha)\|_2^2 & 0 \\ 0 & \|P_+(\overline{h_\alpha})\|_2^2 \end{array} \right] + \mathcal{O}(\varepsilon^3);$$

where $h_\alpha = \alpha 1_{Q_1} - \overline{\alpha} 1_{Q_2} - \alpha 1_{Q_3} + \overline{\alpha} 1_{Q_4}$.

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Assume by contradiction that $\widetilde{Z^{\alpha,\varepsilon}}[0] = \ell_2^N$ for any $\alpha \in \mathbb{T}$ and small ε. Then we must have $\|P_+(h_\alpha)\|_2^2 = \frac{1}{2}$ for any $\alpha \in \mathbb{T}$. In particular,

$$\alpha \mapsto \|P_+(h_\alpha)\|_2^2 \text{ is a constant function on } \mathbb{T}.$$

It follows that the following function is a constant function:

$$\begin{aligned}
C(\alpha) &= \Re\langle \alpha P_+(1_{Q_1}), -\overline{\alpha}P_+(1_{Q_2})\rangle + \Re\langle \alpha P_+(1_{Q_1}), \overline{\alpha}P_+(1_{Q_4})\rangle \\
&+ \Re\langle -\overline{\alpha}P_+(1_{Q_2}), -\alpha P_+(1_{Q_3})\rangle + \Re\langle -\alpha P_+(1_{Q_3}), \overline{\alpha}P_+(1_{Q_4})\rangle.
\end{aligned}$$

Clearly, by translation invariance of Haar measure, we have

$$\langle P_+(1_{Q_1}), P_+(1_{Q_2})\rangle = \langle P_+(1_{Q_2}), P_+(1_{Q_3})\rangle = \langle P_+(1_{Q_3}), P_+(1_{Q_4})\rangle,$$

$$\langle P_+(1_{Q_1}), P_+(1_{Q_4})\rangle = \langle P_+(1_{Q_2}), P_+(1_{Q_1})\rangle,$$

hence

$$C(\alpha) = -\Re\Big\{2\alpha^2\Big(\langle P_+(1_{Q_1}), P_+(1_{Q_2})\rangle - \overline{\langle P_+(1_{Q_1}), P_+(1_{Q_2})\rangle}\Big)\Big\}.$$

Then $\alpha \mapsto C(\alpha)$ is constant function if and only if

$$\langle P_+(1_{Q_1}), P_+(1_{Q_2})\rangle - \overline{\langle P_+(1_{Q_1}), P_+(1_{Q_2})\rangle} = 0,$$

which is equivalent to

(23) $$\Im\Big(\langle P_+(1_{Q_1}), P_+(1_{Q_2})\rangle\Big) = 0.$$

By a similar computation as in the proof of inequality (18), we have

$$\Im\Big(\langle P_+(1_{Q_1}), P_+(1_{Q_2})\rangle\Big) = \frac{4}{\pi^2}\sum_{k=0}^{\infty}\frac{2k+1}{(4k+1)^2(4k+3)^2},$$

this contradicts (23), and hence completes the proof. $\qquad\square$

6 Appendix

Here we reformulate the argument of [CCRSW82] to emphasize the crucial role played by a certain inner function associated to the measurable partition of the unit circle in proving Theorem 1.1. It follows from the preceding that the analogous inner function for a measurable partition into 3 subsets does not exist.

Lemma 6.1. *Suppose that $\Gamma_0 \cup \Gamma_1$ is a measurable partition of $\mathbb{T}$. Then there exists an inner function φ such that $\varphi(0) = 0$. And $\varphi(\Gamma_0) \cup \varphi(\Gamma_1)$ is a partition of $\mathbb{T}$ into two disjoint arcs (up to negligible sets). Moreover,*

(24) $$m(\varphi(\Gamma_0)) = m(\Gamma_0) \text{ and } m(\varphi(\Gamma_1)) = m(\Gamma_1).$$

Proof. Since any origin-preserving inner function φ preserves the measure m on $\mathbb{T}$ (indeed note $\int_{\mathbb{T}} \varphi(\gamma)^n dm(\gamma) = \int_{\mathbb{T}} \gamma^n dm(\gamma) \,\forall n \in \mathbb{Z}$), it suffices to show the existence of an inner function satisfying the partition condition.

Let $v = 1_{\Gamma_1} : \mathbb{T} \to \mathbb{R}$ be the characteristic function of Γ_1, its harmonic extension on $\mathbb{D}$ will also be denoted by v. Note that $0 < v(z) < 1$ for any $z \in \mathbb{D}$. Let $\tilde{v}$ be the harmonic conjugate of v and define $\psi = v + i\tilde{v}$ on $\mathbb{D}$. Then ψ is an analytic map from $\mathbb{D}$ to $\mathcal{S} := \{z \in \mathbb{C} : 0 < \Re(z) < 1\}$ and has non-tangential limit $\psi(\gamma) = v(\gamma) + i\tilde{v}(\gamma)$, a.e. $\gamma \in \mathbb{T}$. Thus

$$\psi(\Gamma_0) \subset \partial_0 \text{ and } \psi(\Gamma_1) \subset \partial_1,$$

where $\partial_0 = \{z \in \mathbb{C} : \Re(z) = 0\}$ and $\partial_1 = \{z \in \mathbb{C} : \Re(z) = 1\}$. Let $\tau : \mathcal{S} \to \mathbb{D}$ be a Riemann conformal mapping such that $\tau(\psi(0)) = 0$. Note that $\tau(\partial_0)$ and $\tau(\partial_1)$ are disjoint open arcs of $\mathbb{T}$. Define $\varphi = \tau \circ \psi : \mathbb{D} \to \mathbb{D}$. Then φ is an inner function such that $\varphi(0) = 0$. We have

$$\varphi(\Gamma_0) \subset \tau(\partial_0) \text{ and } \varphi(\Gamma_1) \subset \tau(\partial_1).$$

Hence $m(\varphi(\Gamma_0)) \leq m(\tau(\partial_0))$ and $m(\varphi(\Gamma_1)) \leq m(\tau(\partial_1))$. Since φ preserves the measure m, we have

$$1 = m(\Gamma_0) + m(\Gamma_1) = m(\varphi(\Gamma_0)) + m(\varphi(\Gamma_1)) \leq m(\tau(\partial_0)) + m(\tau(\partial_1)) = 1.$$

Thus up to negligible sets, we have

$$\varphi(\Gamma_0) = \tau(\partial_0) \text{ and } \varphi(\Gamma_1) = \tau(\partial_1).$$

$\square$

Remark 6.2. *Implicitly, we have used the fact that if $\varphi : \mathbb{T} \to \mathbb{T}$ is measure preserving, then the image $\varphi(A)$ of any measurable set A is measurable. It is an easy consequence of the characterization of measurability (coincidence of interior and exterior measures) and Egorov's theorem.*

Proof of Theorem 1.1. Suppose $\Gamma_0 \cup \Gamma_1$ is a measurable partition of the circle and let the interpolation family $\{X_\gamma : \gamma \in \mathbb{T}\}$ be such that

$$X_\gamma = Z_0 \text{ for all } \gamma \in \Gamma_0, \ X_\gamma = Z_1 \text{ for all } \gamma \in \Gamma_1.$$

By Lemma 6.1, we can find an inner function φ such that $\varphi(0) = 0$ and $\varphi(\Gamma_0) = J_0$, $\varphi(\Gamma_1) = J_1$ up to negligible sets, where $J_0 \cup J_1$ is a partition of the circle into disjoint arcs. Consider the interpolation family of spaces $\{\widetilde{X}_\gamma : \gamma \in \mathbb{T}\}$ such that

$$\widetilde{X}_\gamma = Z_0 \text{ for all } \gamma \in J_0, \ \widetilde{X}_\gamma = Z_1 \text{ for all } \gamma \in J_1.$$

Then by a conformal mapping, it is easy to see

(25)
$$\widetilde{X}[0] = (Z_0, Z_1)_\theta, \ \theta = m(J_1) = m(\Gamma_1).$$

We have $\widetilde{X}_{\varphi(\gamma)} = X_\gamma$ for a.e. $\gamma \in \mathbb{T}$. If $x \in \mathbb{C}^N$ is such that $\|x\|_{\widetilde{X}[0]} < 1$, then by definition, there exists an analytic function $f : \mathbb{T} \to \mathbb{C}^N$ such that $f(0) = x$ and $\operatorname*{ess\,sup}_{t \in \mathbb{T}} \|f(\gamma)\|_{\widetilde{X}_\gamma} < 1$. Thus

$$\operatorname*{ess\,sup}_{\gamma \in \mathbb{T}} \|(f \circ \varphi)(\gamma)\|_{X_t} = \operatorname*{ess\,sup}_{\gamma \in \mathbb{T}} \|(f \circ \varphi)(\gamma)\|_{\widetilde{X}_{\varphi(\gamma)}} = \operatorname*{ess\,sup}_{\gamma \in \mathbb{T}} \|f(\gamma)\|_{\widetilde{X}_\gamma} < 1.$$

Since $(f \circ \varphi)(0) = f(0) = x$, the above inequality shows that $\|x\|_{X[0]} < 1$. By homogeneity, $\|x\|_{X[0]} \leq \|x\|_{\widetilde{X}[0]}$. But if we consider the dual of the above interpolation family, then we get the same inequality, hence we must have

(26)
$$\|x\|_{X[0]} = \|x\|_{\widetilde{X}[0]}.$$

By (26) and (25), we have

$$X[0] = (Z_0, Z_1)_\theta, \ \theta = m(\Gamma_1).$$

$\square$

Remark 6.3 (Communicated by Gilles Pisier). *The preceding argument also shows that, as conjectured in [Pis10], of which we use the terminology, any θ-Hilbertian Banach space is automatically arcwise θ-Hilbertian, at least under suitable assumptions on the dual spaces, that are automatic in the finite dimensional case. We merely indicate the argument in the latter case. Consider a measurable partition $\Gamma_0 \cup \Gamma_1$ of the unit circle with $m(\Gamma_1) = \theta$ and a family of n-dimensional spaces $\{E_\gamma \mid \gamma \in \partial D\}$ such that $E_\gamma = \ell_2^n$ for any $\gamma \in \Gamma_1$ but E_γ is arbitrary for $\gamma \in \Gamma_0$. If φ is the inner function appearing in Lemma 6.1, and if we set $F_\gamma = E_{\varphi(\gamma)}$ then the identity map $Id : E[0] \to F[0]$ is clearly contractive and $F[0]$ is arcwise θ-Hilbertian. Applying this to the dual family $\{E_\gamma^*\}$ in place of $\{E_\gamma\}$ and using the duality theorem from [CCRSW79, Th. 2.12]) we find that $Id : E[0]^* \to F[0]^*$ is also contractive, and hence is isometric. This shows that $E[0]$ is arcwise θ-Hilbertian.*

Acknowledgements

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